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Black Holes, Entropy and Resolving a Paradox

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Abstract

This dissertation covers some of the ideas needed to understand what the black hole information paradox is, and some recent progress in providing a mechanism for accounting for the supposedly lost information. We will briefly visit the AdS/CFT correspondence and see how this can be used to calculate entanglement entropy in general, and then in the black hole context. Finally we will briefly see how these ideas have been applied to a simple model of a black hole to reproduce unitary evolution.

Acknowledgments

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Contents

Chapter 1

Introduction

The Black hole Information paradox, is no longer a paradox, or at least is closer than ever to not being one. Our understanding of black holes has been a driving force in our goal to understand how quantum mechanics and gravity can be reconciled. In this dissertation I will aim to review much of the necessary information required to understand what the problem is, and how recent progress has aimed to solve the paradox. Our first job will be to understand how black holes give off radiation. This was first understood in Hawking's original 1975 paper [1]. This was after the discovery that black holes follow a set of laws analogous to the standard laws of thermodynamics, black hole entropy will be a key player. Namely the Bekinstein-Hawking entropy $S_{BH} = \frac{A}{4G_N}$ which will be a starting point in finding a fine grained formula for the black hole entropy.

We will then move on to look at quantum field theory in curved space-time. This is a nice approach as you can gain an understanding about black hole evaporation without ever mentioning them. Our main study will be the Unruh effect, uncovered by Fulling in 1973 [2]. We will also see a way of deriving hawking radiation directly from the Schwarzschild black hole metric.

Our treatment of QFT in curved spacetime will consist of generalising covariant quantisation when we have a metric that is dynamical. This will mean that our notion of what a particle is, will change depending on the metric. In more technical terms, we will redefine the raising and lowering operators and find the corresponding transformation between them. From this we find that we cannot define a unique vacuum relative to all observers, even in flat spacetime. The Unruh effect is the statement that a uniformly accelerated observer in Rindler coordinates will see the vacuum as thermal and hence observe a temperature. Using simple arguments from this and the fact that the near horizon coordinates of a Schwarzschild black hole looks like Rindler coordinates we can deduce that black holes have a temperature given by $T_H = \frac{\hbar\kappa}{2\pi k_B}$.

Chapter 4 is an introduction to the notion of entanglement entropy and how taking a black hole to work as a quantum system we can predict the entropy of that system will act as follows. Page proposes [3, 4] that if a black hole is viewed as a quantum bipartate system, formed from a pure state, then the entanglement entropy of the radiation follows the so called "Page curve". The main aim of many modern works tackling the information paradox is reproducing this curve in the entropy of the Hawking radiation. The main trouble is, the absence of full theory of quantum gravity. We focus on methods of calculating the entropy of subsystems in Conformal field theories, such as the "Replica trick", which stars in recent progress on the black hole problem [5].

Next we introduce the AdS/CFT correspondence, this is a conjectured equivalence between specific quantum field theories, with additional conformal symmetry, and gravitational theories in AdS. The AdS/CFT correspondence provides some respite if we don't like calculating Entanglement entropies (EEs) in the language of Conformal Field Theory (CFT). As we saw in the previous section, calculating EEs directly from the CFT language gets very tricky very fast. We take the same example of the single interval and calculate the result equivalently without the replica trick in AdS_3.

In the next section we introduce the Ryu-Takayanghi (RT) formula that conjectures a link between entropies of CFTs, and extremal surfaces in AdS space. Subsequent higher order corrections to this formula were made by Faulkner, Lewkowycz and Maldacena (FLM) [6], then Engelhardt and Wall (EW) [7]. These led us to the most recent leap in progress [5, 8] to include regions behind the horizon dubbed ”Islands” in the calculation of S_{rad}. These regions begin to appear some time after the black hole has formed and have been shown, in a toy model of a black hole, to unitary evolution of the original state, in this case will be a levelling out of the Page curve.

Chapter 2

Background

2.1 Black hole thermodynamics

In this chapter we will outline some of the essential tools, results and ideas used to understand black holes as thermodynamic objects. This chapter will be mainly based on lecture notes by Dowker [9], Townsend [10], and Gauntlett [11]. Starting with a simple black hole background we can derive equations that look suspiciously like the laws of thermodynamics from standard statistical mechanics. This was originally stated in Carter and Hawking's 1973 paper [12] where they layed down *The four laws of black hole mechanics* and [13]. First we must make some definitions.

Definition:Null hypersurface

Let $S(x)$ be a smooth function of our spacetime coordinates x^μ and consider the family of hypersurfaces $S = constant$. The vector fields normal to this surface are

$$l = \tilde{f}(x)(g^{\mu\nu}\partial_\nu S)\frac{\partial}{\partial x^\mu} \tag{2.1}$$

Where $\tilde{f}$ is an arbitrary non-zero function. If $l^2 = 0$ for any hypersurface $\mathcal{N}$ in this family of surfaces. Then $\mathcal{N}$ is a null hypersurface. So vectors normal to null hypersurfaces, are also tangent to the surface.

Definition: Killing vector

A vector field $\xi^\mu(x)$ is a killing vector if

$$\mathcal{L}_\xi g = 0 \tag{2.2}$$

Definition:Killing horizon

A null hypersurface $\mathcal{N}$, is a Killing horizon of a killing vector ξ^μ if ξ^μ is normal to $\mathcal{N}$ on $\mathcal{N}$.

Definition:Surface gravity

On $\mathcal{N}$ there exists a function κ such that

$$\nabla_\mu \xi^\nu \xi_\nu |_{\mathcal{N}} = -2\kappa \xi_\mu \tag{2.3}$$

κ is called the *surface gravity*. The surface gravity will be interpreted to be related to the temperature of the black hole.

2.2 The laws of black hole mechanics

In 1973 Hawking published a paper [12] relating the way black holes change when mass is added, to the laws of thermodynamics from standard statistical mechanics.

- **Zeroth law**

  The surface gravity κ is constant over the event horizon of a stationary black hole. This law draws a comparison between a black hole and any thermodynamic system in thermal equilibrium, where temperature is constant everywhere.

- **First law**

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ \tag{2.4}$$

Where M is the mass, A is the area of the horizon, J is the angular momentum, and Ω_H is the rotational velocity of the horizon. This law is derived directly from the metric by considering first the area of the horizon, and considering a small variation in the parameters M and a. We first find the area of the horizon by setting $r = r_+ = M + \sqrt{M^2 - a^2}$ (working in units where $G = 1$) and we get an induced metric on the horizon [14].

$$ds^2_{\text{horizon}} = (r_+^2 + a^2 cos^2\theta)d\theta^2 + \left[\frac{(r_+^2 + a^2)^2 sin^2\theta}{r_+^2 + a^2 cos^2\theta}\right]d\phi^2 \tag{2.5}$$

To find the area we integrate to induced volume element from the new metric. We find that

$$A = 4\pi(r_+^2 + a^2) \tag{2.6}$$

The law comes directly from varying the area (after a lot of algebra, of course). The analogy with traditional thermodynamics comes from assuming that the entropy is proportional to the area of the horizon.

- **Second law**

$dA \geq 0$ This states that when two black holes collide, the area of the resulting black hole is greater than the sum of the two areas. This is reminiscent of the second law of thermodynamics that states that the entropy always increases. This is suggestive of a link between the area and the entropy of the black hole.

- **Third law** The surface gravty κ cannot be reduced to zero in a finite number of operations.

These laws are for the moment purely classical, they can be derived directly from general relativity and Einstein's equations. Despite these suggestive equations, It was not believed that black holes were thermodynamic objects, until 1975 when Hawking discovered that black holes, aren't actually completely black. They give off radiation, and have a temperature proportional to the surface gravity

$$T_H = \frac{\hbar\kappa}{2\pi} \tag{2.7}$$

The generalised entropy formula for a black hole is

$$S_{gen} = \frac{A}{4\hbar G_N} + S_{outside} \tag{2.8}$$

This will turn out to be the wrong formula and lead to a monotonically increasing entropy as the black hole evaporates, which seems to create entropy out of nothing. This is bad for quantum mechanics, since a black hole formed from a pure state, which should have no entropy, seems to unitarily evolve into a mixed state!

Chapter 3

QFT in curved space-time

This chapter will start with the general formulation of QFT in curved space-time and show some important phenomena that occur when we allow our background metric to be dynamical and our definition of what a ”particle” is to change with it. This will then lead us to considering how our QFT changes when we simply change the coordinate system (in our case this will be Rindler co-ordinates) and give a key link to understanding how black holes evaporate.

3.1 Scalar field quantisation

This short introduction to the formulation of QFT in curved spacetime is largely based on Birrel and Davies [15] and Fay Dowkers notes [9]. To go from QFT in flat spacetime, to curved, the first step must be to express our equations in a coordinate invariant form. We will focus on the simple case of the real Klein-Gordon scalar. We start with the Lagrangian density..

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2) \tag{3.1}$$

Where $\phi = \phi(t, \boldsymbol{x})$ This leads to the equation of motion.

$$\nabla^2\phi - m^2\phi = 0 \tag{3.2}$$

And conjugate momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial(\nabla_0\phi)} = \sqrt{-g}\nabla_0\phi \tag{3.3}$$

We now can impose the canonical commutation relations (as usual)

$$\left[\hat{\phi}(t, \boldsymbol{x}), \hat{\phi}(t, \boldsymbol{y})\right] = 0 \tag{3.4}$$

$$\left[\hat{\pi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})\right] = 0 \tag{3.5}$$

$$\left[\hat{\phi}(t, \boldsymbol{x}, \hat{\pi}(t, \boldsymbol{y})\right] = \frac{i}{\sqrt{-g}}\delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{3.6}$$

Our next step will be to define an inner product.

$$(\phi_1, \phi_2) = -i\int_\Sigma d\Sigma^\mu\sqrt{-g}(\phi_1\partial_\mu\phi_2^* - \phi_2\partial_\mu\phi_1^*) \tag{3.7}$$

Where we take Σ to be a Cauchy surface, under the assumption that our spacetime is globally hyperbolic (for precise definitions of these terms see [16]). The inner product is invariant of the choice of Cauchy surface.

Now, there exist some $u_i(x)$ such that they are a complete set of mode solutions, with orthonormal relations in the product (3.7).

$$(u_i, u_j) = \delta_{ij}, \qquad (u_i^*, u_j^*) = -\delta_{ij}, \qquad (u_i, u_j^*) = 0 \tag{3.8}$$

We can now expand the field ϕ in terms of its mode solutions and some raising and lowering operators a_i and their conjugates a_i^*

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] \tag{3.9}$$

Now impose commutation relations on these operators.

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \tag{3.10}$$

We now have the setup required to build a fock space in the usual way. By defining the vacuum state $|0\rangle$ to be annihilated by lowering operators a_i. We form states by acting on the vacuum state with successions of raising operators $a_{i_n}^\dagger$. Doing this we form a basis for a Hilbert space H.

However there are some caveats when working in curved spacetime. The way we expand the field $\phi(x)$ in terms of raising and lowering operators and mode solutions is certainly not unique, so we must think of a way to account for this ambiguity. Note that this is already hinting at the Unruh effect, in that we seem to be unable to define a unique vacuum state. So the next step is to question how we relate different expansions of the field $\phi(x)$.

3.1.1 Bogoliubov transformations

Say $\phi(x)$ can be expanded in some other way.

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)] \tag{3.11}$$

Where we now have different raising and lowering operators that now by definition have to kill a new vacuum, so that.

$$\bar{a}_j \left|\bar{0}\right\rangle = 0, \forall j \tag{3.12}$$

Since both are complete sets of modes, complete meaning that they satisfy the correct set of orthonormality conditions, we can express $\bar{u}_j$ in terms of the old modes with some relating coefficients. We call these Bogoliubov coefficients.

$$\bar{u}_j = \sum_i [\alpha_{ji} u_i + \beta_{ji} u_i^*] \tag{3.13}$$

$$\bar{u}_j^* = \sum_i [\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i] \tag{3.14}$$

Now we shall impose our orthonormality conditions on the new model solutions and see some properties of the Boguliubov coefficients.

By demanding (3.8) we get

$$\alpha\alpha^\dagger - \beta\beta^\dagger = I \tag{3.15}$$

$$\alpha\beta^T - \beta\alpha^T = 0 \tag{3.16}$$

The coefficients are calculated using

$$\alpha_{ij} = (\bar{u}_i, u_j), \qquad \beta_{ij} = -(\bar{u}_i, u_j^*) \tag{3.17}$$

Now we have the necessary relations in order to express the new raising and lowering operators in terms of the old ones. Our trajectory from here is beginning to get clearer. We are going the end up with a different expression for the new a_i operators and therefore the number operator $\hat{N} = a_i^\dagger a_i$ will have a different vacuum expectation value.

our expansion for $\bar{a}_i$ in terms of the Boguluibov coefficients is.

$$\bar{a}_i = \sum_i [\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger] \tag{3.18}$$

3.1.2 Particle production in dynamical spacetimes

We now have ther tools to think about how particles are defined in spacetimes that change in time. In Minkowski spacetime, or even a more exotic one that does not change, we have a universal notion of a particle. When curvature is non-stationary, we have a changing definition of a one particle state. To see this, let's consider the so-called "sandwich spacetime" $M = M_- \cup M_0 \cup M_+$. Where in M_- we expand the scalar field in solutions of the Klein-Gordon equation. When we evolve to M_+ the functions used in M_- will no longer solve the Klein-Gordon equation so we expand in different functions to represent the same field. We must now use our Bogoliubov transformations to relate M_- and M_+.

We showed before, the transformed number operator with respect to the old raising and lowering operators, is given by $\bar{N}_i = \bar{a}_i^\dagger \bar{a}_i$. Which we shall calculate it's expecta-

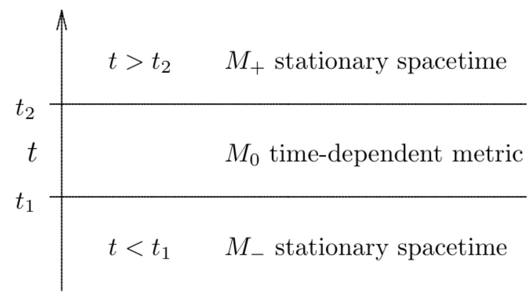

Figure 3.1: The sandwich spacetime. This is essentially a nice way of visualising a dynamical spcetime. The field expansion will change between time t_1 to t_2 [10]

tion value in the original vacuum. The original a_i's kill the original vacuum $|0\rangle$ and we are left with a non-zero expectation value of the number operator in the vacuum.

$$\langle 0|\, \bar{N}_i \,|0\rangle = \langle 0|\, \bar{a}_i^\dagger \bar{a}_i \,|0\rangle \tag{3.19}$$

$$= \sum_{jk} \langle 0|\, a_k(-\beta_{ik})(-\beta_{ij}^* a_j^\dagger) \,|0\rangle \tag{3.20}$$

$$= \sum_{jk} \langle 0|\, a_k a_j^\dagger \,|0\rangle\, \beta_{ik}\beta_{ij}^* \tag{3.21}$$

$$= \sum_{j} \beta_{ij}\beta_{ij}^* \tag{3.22}$$

$$= (\beta\beta^\dagger)_{ii} = \mathrm{Tr}\big(\beta\beta^\dagger\big) \tag{3.23}$$

Note that there is no summation over the index here. So we get an expectation value that is not necessarily zero. Therefore if β is non zero we get particle production in our time dependant spacetimes!

3.2 The Unruh effect

We will next explore a stunning result from QFT in curved spacetime first predicted by Fulling in 1973 [2] and subsequently Davies [17] and Unruh [18]. The basic idea of the Unruh effect is similar to what we have already covered. We can observe a temperature from the vacuum, in this example we will explicitly show how this effect can even be observed in flat spacetime.

Consider for simplicity, two-dimensional Minkowski spacetime with the metric.

$$ds^2 = dt^2 - dx^2 \tag{3.24}$$

Our task will be to consider the massless wave equation on this metric and transform it into a uniformly accellrated coordinates (Rindler space). We will then need to find the Bogoliubov coefficients in order explicitly calculate the vacuum expectation value in these coordinates.

First we choose new coordinates (η, ξ) which reflect the perspective of a uniformly accelerated observer. Where $-\infty < \eta, \xi < \infty$

$$t = \frac{1}{a}e^{a\xi}\sinh a\eta \tag{3.25}$$

$$x = \frac{1}{a}e^{a\xi}\cosh a\eta \tag{3.26}$$

Or in null coordinates $(\bar{u}, \bar{v})$

$$\bar{u} = -\frac{1}{a}e^{-au} \tag{3.27}$$

$$\bar{v} = \frac{1}{a}e^{av} \tag{3.28}$$

This leads to the new metric. where a is These coordinates now describe the right Rindler wedge, shown in fig.(3.2).

Changing coordinates again with $u = \eta - \xi$, $v = \eta + \xi$ (3.27) becomes.

$$ds^2 = e^{2a\xi}dudv = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{3.29}$$

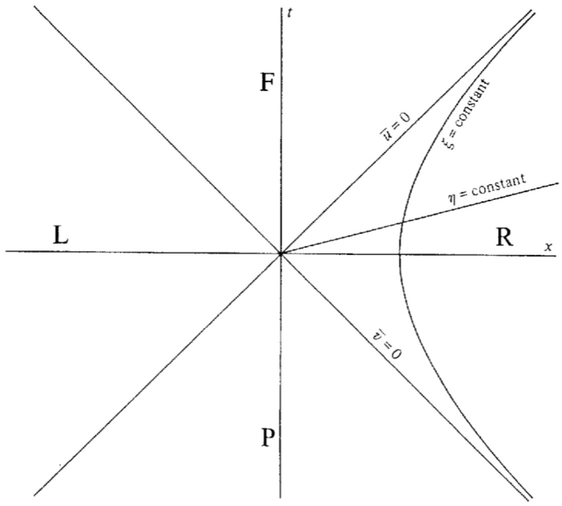

Figure 3.2: Representation of rindler spacetime, the hyperbolae represent the paths taken by uniformly accelerated observers. Figure taken from Birrell and Davies [15]

The Rindler spacetime is now split into four causally disconnected regions. The lines $\bar{u} = 0$ and $\bar{v} = 0$ act as event horizons. where a uniformly accelerated observer will not cross between wedges. When we consider quantisation on this spacetime we must include solutions for both the left and the right Rindler wedge. The left wedge is simply defined by reversing the signs in our original transformation (3.25)-(3.26)

Consider our massless wave equation in Minkowski and then Rindler coordinates.

$$\Box\phi = e^{-2s\xi}(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2})\phi = e^{-2a\xi}\frac{\partial^2\phi}{\partial u\partial v} = 0 \tag{3.30}$$

We can very easily solve this with a complete set of positive frequency mode solutions. The modes on the right wedge however, will not be considered positive frequency in the left wedge. To deal with this we take two sets of solutions. Firstly in Minkowski space.

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}}e^{ikx-i\omega t} \tag{3.31}$$

Where $\omega =\mid k \mid$ and $-\infty < k < \infty$.

And in Rindler coordinates.

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi \pm i\omega\eta} \tag{3.32}$$

$$\omega =\mid k, \qquad -\infty < k < \infty \tag{3.33}$$

Where the positive sign on the η term is associated with the positive frequency solutions on the right Rindler wedge. vice versa for the negative sign. We represent this as two separate mode solutions.

$$^{R}u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - i\omega\eta}, & \text{in R} \\ 0, & \text{in L} \end{cases} \tag{3.34}$$

$$^{L}u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi + i\omega\eta}, & \text{in L} \\ 0, & \text{in R} \end{cases} \tag{3.35}$$

Now we have all of our solutions in the left and right wedge, we can expand it in the usual way and define what we think a particle is, as a Rindler observer. Each solution and its conjugate will be associated with a different raising operator for these "Rindler particles". Note that we can do this because the left and right moving modes form a complete basis for functions on the manifold.

$$\phi = \sum_{k=-\infty}^{\infty} (b_k^{(1)L} u_k + b_k^{(1)\dagger L} u_k^* + b_k^{(2)R} u_k + b_k^{(2)\dagger R} u_k^*), \tag{3.36}$$

Expressed in minkowski coordinates would simply be.

$$\phi = \sum_{k=-\infty}^{\infty} (a_k \bar{u}_k + a_k^\dagger \bar{u}_k^*) \tag{3.37}$$

Now we have the field expressed in both coordinate systems we can begin to build two separate fock spaces based off the two vacuums $|0_M\rangle$ and $|0_R\rangle$. Both defined in the standard way, by demanding that they are killed by their respective annihilation operators. In the Rindler case, $|0_R\rangle$ is killed by both its left and right wedge annihilation operator.

Our next step will be to relate the two solutions via our Bogoliubov coefficients (3.17). Our calculation will follow Dowkers treatment in [9].

Recall (3.13) we get the right handed positive frequency Rindler modes.

$$u_j^R = \int_0^\infty d\omega' [\alpha_{\omega\omega'} u_{\omega'} + \beta_{\omega\omega'} u^*_{\omega'}] \tag{3.38}$$

Note the Minkowski modes take the form $u_{\omega'} = \frac{1}{\sqrt{4\pi\omega'}} e^{-i\omega'(x-t)}$, where we are focusing on the right moving modes $k > 0$ so $\omega = k$. And from now on for ease of notation we will define $v = x - t$

Next we define the fourier transform of the right moving Rindler mode

$$u_\omega^R(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' e^{-i\omega' v} \tilde{u}_\omega(\omega') \tag{3.39}$$

where

$$\tilde{u}_\omega(\omega') = \int_{-\infty}^{\infty} d\bar{v} e^{i\omega'\bar{v}} u_\omega^R(\bar{v}) \tag{3.40}$$

We split up (3.39) in the following way, our aim at the moment is to find the β coefficient.

$$u_\omega^R(\bar{v}) = \frac{1}{2\pi} \int_0^\infty d\omega' e^{-i\bar{v}\omega'} \tilde{u}_\omega(\omega') + \frac{1}{2\pi} \int_0^\infty d\omega' e^{i\bar{v}\omega'} \tilde{u}_\omega(-\omega') \tag{3.41}$$

We then compare this with our original expression for the mode expressed in terms of Bogoliubov coefficients. We find that

$$\alpha_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{u}_\omega(\omega') \qquad \beta_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{u}_\omega(-\omega') \tag{3.42}$$

We now must use the following relation which we will state but not prove, for a proof see [9]. We claim that $\tilde{u}_\omega(-\omega') = e^{-\frac{-\pi\omega}{a}}\tilde{u}_\omega(\omega')$. This is useful when used in conjunction with the relation (3.15). We can find an expression for $|\beta_{\omega\omega}|^2$.

We evaluate the Rindler number operator in the Minkowski vacuum to obtain the ”Unruh temperature”.

$$\langle 0_M| \, b_k^{(1,2)\dagger} b_k^{(1,2)} \, |0_M\rangle = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \tag{3.43}$$

Which is the planck spectrum with at a temperature.

$$T = \frac{a}{2\pi k_B} \tag{3.44}$$

Where k_B is the Boltzman constant.

This is an amazing result. Even in flat spacetime, we predict that a uniformly accelerated observer will see the Minkowski vacuum as thermal.

3.3 Hawking radiation

If we start with the usual metric for a Schwarzschild geometry

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r})} + r^2 d\Omega^2 \tag{3.45}$$

Once again we go to near horizon coordinates by defining

$$r = 2GM(1 + \frac{\rho^2}{4(2GM)^2}) \quad t = 2(2GM)T \tag{3.46}$$

Following a similar procedure to that in (2.2.1) and taking the limit $\rho << 2GM$ we arrive at the near horizon metric

$$ds^2 = -\rho^2 dT^2 + d\rho^2 + \text{higher order terms} \tag{3.47}$$

Which is the metric for Rindler space. This is where our hard work on the Unruh effect comes to fruition. an observer near the horizon will be uniformly accelerating and will observe a temperature. This is an easy way to convince yourself that black holes have a temperature if you can accept the Unruh effect. Hawking's calculation showing particle production by black holes requires slightly more work but the basic concept is similar. We will want to find Bogoluibov coefficients and calculate the expectation value of N [1, 9].

Note the Carter-Penrose diagram for a spherically symmetric collapsing star in fig

We must attempt to solve the vaccum Klein-Gordon equation in Schwarzschild space-time.

$$ds^2 = (1 - \frac{2M}{r})(-dt^2 + dr_*^2) + r^2 d\Omega_2^2 \tag{3.48}$$

Using the relation that

$$\Box = \nabla^\mu \nabla_\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \tag{3.49}$$

We will use a spherical harmonic decomposition of the field, namely $\phi(t, r_*, \theta, \phi) =$

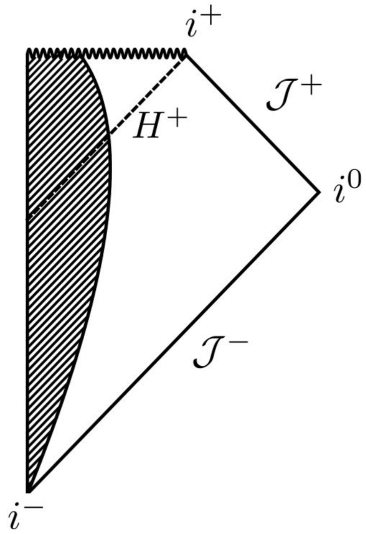

Figure 3.3: Conformal diagram for a spherically symmetric collapsing star, figure from [9].

$\chi_l(r,t)Y_{lm}(\theta,\phi)$, we derive the result

$$[\partial_t^2 - \partial_{r_*}^2 + V_l(r_*)]\chi_l = 0 \tag{3.50}$$

Our aim is to compare the modes on $\mathcal{J}^+$ and $\mathcal{J}^-$. There is a caveat in finding a Cauchy surface on this spacetime as it is not globally hyperbolic, as the asymptotic past near $\mathcal{J}_-$ is a Cauchy surface, so we are fine simply finding positive frequency modes here. $\mathcal{J}^+$ is not a Cauchy surface, however $H^+ \cup \mathcal{J}^+$ is, so we can specify our complete set as a union of modes with support on only $\mathcal{J}^+$ and then H^+ respectively.

In analogy with the Unruh effect, we will want to express the outgoing late modes on $\mathcal{J}^+ \cup H^+$ in terms of the early modes on $\mathcal{J}^-$. We do this by tracing a solution back in time, along a null geodesic and find that part of the mode is transmitted to $\mathcal{J}^-$ and part falls into the collapsing matter. When we start with a mode such as

$g_\omega = e^{-i\omega u}$, we end up with the modes on $\mathcal{J}^-$ looking like

$$g_\omega^{\text{transmitted}} = \begin{cases} e^{\frac{i\omega}{\kappa}\ln(-v)} & \text{for} \quad v < 0 \\ 0 & \text{for} \quad v > 0 \end{cases} \tag{3.51}$$

For these modes our Bogoliubov coefficients will look exactly analogous to the Rindler ones. Except with κ swapped with a. Hereby we can derive the hawking temperature in the same way as we did the Unruh with.

$$T_H = \frac{\hbar\kappa}{2\pi k_B} \tag{3.52}$$

3.3.1 Extremal Reissner-Nördstrom solution

Let's focus on a specific solution to Einstein's equations. The Reissner-Nördstrom solution for a charged black hole. The metric is [19]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{3.53}$$

Where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

As seen in the previous section, we often get interesting results by a change of coordinates. Again, our coordinate change will be near horizon limit. When the black hole is not extremal, we should simply reproduce the previous result of $Rindler \times S^2$. However if we consider the *extremal limit* where $M = Q$, we find a few things happen. Firstly we find that the inner and outer horizons coincide and as a result the black hole no longer radiates.

Next take near horizon coordinates by defining

$$r = Q(1 + \frac{\lambda}{z}), \quad t = \frac{QT}{\lambda} \tag{3.54}$$

And noting that

$$dr^2 = \frac{Q^2\lambda^2}{z^4}dz^2 \tag{3.55}$$

$$dt^2 = \frac{Q^2}{\lambda^2}dT^2 \tag{3.56}$$

$$f(r) = (\frac{\lambda}{z + \lambda})^2 \tag{3.57}$$

Using this, and taking the limit $\lambda \to 0$ we arrive at the metric

$$ds^2 = \frac{Q}{z^2}(-dT^2 + dz^2) + Q^2 d\Omega_2^2 \tag{3.58}$$

This metric represents the geometry $AdS_2 \times S^2$. This will be crucial later on when we consider the AdS/CFT correspondence to calculate the generalised entropy of the black hole.

Chapter 4

Entropy

This aim of this chapter will be to introduce the notion of entropy in a quantum system.We start with some classical information theory as the notion of entropy is easier to understand here and we can make good analogy with the quantum case, we then move on the the different notions of entropy in a quantum system. Most notably we will be interested in the von Neumann entropy, sometimes called the ”fine-grained entropy” of a quantum system. Next we will be concerned with trying to calculate this entropy, more specifically when we have a subsystem of a larger Hilbert space, we call this the entanglement entropy. Later on we will want to view a black hole simply as a quantum system split into two parts, the black hole, and the radiation and our task will be to compute the entanglement entropy of the Hawking radiation.

4.1 Quick review of classical information theory

We will start with the simpler case of classical information theory. Most of the definitions in this section will carry over and be extremely useful in the case of calculating quantum entropy. We will follow Headricks treatment [20]

Take some probability distribution

$$p_a \geq 0 \qquad \sum_a p_a = 1 \tag{4.1}$$

From a simplistic point of view, the case of lowest entropy, were we know the most about any outcome would be a determanistic distribution, where $p_{a_i} = 1$ for some $1 \leq i \leq n$, this would correspond to S=0. On the other hand maximum entropy would correspond to a uniform distribution where $p_{a_i} = \frac{1}{n} \quad \forall i$. The definition of the shannon entropy is

$$S(\boldsymbol{p}) = -\sum_a p_a \ln p_a$$

There are a few importanrt properties of the shannon entropy. Say we have two adjoined systems A and B.

- Extensiveness This means that if we consider two independent systems A and B, the entropy of the system as a whole is simply the sum of the two entropies.
- Subadditivity - $S(AB) \leq S(A) + S(B)$ This is an important condition as it relates to entanglement in the quantum sense. Equality holds if and only if A and B are independent. If they are not then they must be correlated in some way.
- Conditional entropy $H(A|B) = S(AB) - S(B)$ This is the entropy of the system as a whole after learning of the state B.
- Mutual information - $I(A : B) = S(A) - H(A|B)$ This tells you information gained on A, when we learn about B.
- The maximum entropy of any system is $S_{max}(A) = \ln \dim \mathcal{H}_A$.

4.2 The density matrix and von Neumann entropy

Firstly we will want to define an object that encodes all that we know about a quantum system, that is to calculate expectation values of observables. We do this by defining the *density matrix* ρ which is an operator that acts on our Hilbert space H.

$$\rho^\dagger = \rho, \qquad \rho \geq 0, \qquad \operatorname{Tr} \rho = 1 \tag{4.2}$$

The density matrix is expressed as $\rho = \sum_i p_i \left|\psi\right\rangle \left\langle\psi\right|$ where each p_i represents the probability of being in the state i. Pure states are ones that can be expressed in the form $\rho = \left|\psi\right\rangle \left\langle\psi\right|$, any state that cannot be factorised in this way are called mixed.[20]

From the density matrix we can extract a quantity known as the *von Neumann entropy*, it is defined to be.

$$S(\rho) := -\operatorname{Tr} \rho \ln \rho \tag{4.3}$$

This is useful as it satisfies many properties we would expect an entropy to satisfy. The von Neumann entropy of a pure state is zero. This is resonable, seeing as we "know everything about the state of the system". This quantity also has many of the tropes that we expect from classical information theory (Shannon entropy). With an additional nice property that it is invariant under unitary transformations [21]. Another useful property to note is that $\operatorname{Tr}(\rho^2) \leq 1$ with equality if and only if ρ is a pure state.

It will be useful to define the dentisy matrix in a thermal state. it is useful to think of the topology of our theory, for example on a circle. Our density matrix is represented by a path integral with two open cuts, so this would be a cylinder with boundary conditions on the top and bottom. [19].

$$\rho = \frac{1}{Z} e^{-\beta H} \tag{4.4}$$

Where β is the inverse temperature and H is called the *modular Hamiltonian*. Z is the partition function and is there to ensure states are properly normalised. We can represent the trace as a Euclidean path integral over a cylinder with the ends identified, in other words, a torus.

$$Z = \mathrm{Tr}\, e^{-\beta H} \tag{4.5}$$

4.2.1 Rényi entropies

Practically when calculating the von Neumann entropy of an actual system, we go via an an indirect but often much easier route of the Rényi entropy. These are defined as follows [20].

$$S_\alpha := \frac{1}{1-\alpha} \ln \mathrm{Tr}\, \rho^\alpha = \frac{1}{1-\alpha} \ln\left(\sum_a \rho_a^\alpha\right), \qquad (\alpha \neq 0, 1, \infty) \tag{4.6}$$

$$S_0 := \lim_{\alpha \to 0} S_\alpha = \ln(rank \rho) \tag{4.7}$$

$$S_1 := \lim_{\alpha \to 1} S_\alpha = S \tag{4.8}$$

$$S_\infty := \lim_{\alpha \to \infty} S_\alpha := -\ln \|\rho\| = -\ln\Big(\max_a \rho_a\Big) \tag{4.9}$$

Where ρ_a are the eigenvalues of ρ. We will later on outline a specific example that is very important to our story.

4.2.2 Entanglement Entropy

Our next important idea will start with splitting up a quantum system into two spaces, often called a *bipartate system*. Take a Hilbert space, made up of the tensor product between two separate systems A and B.

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{4.10}$$

We can set A to be the smaller system ie. $|A| < |B|$. For any system such as this there will be a density matrix describing it, ρ_{AB}. We define the *reduced density matrix* describing subsystem A as

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB} \tag{4.11}$$

From this we define the entanglement entropy of A as the von Neumann entropy of the reduced density matrix associated to A.

$$S_A = -\,\mathrm{Tr}\, \rho_A \ln \rho_A \tag{4.12}$$

There are many useful properties of von Neuman entropies that we will make use of in the coming chapter, such as subadditivity and extensiveness. So if ρ_{AB} is a product state, then the entropy of the total system is simply the sum of the entropy of each individual system. If not, then there must be some mixture of states, said in symbols.

$$\rho_{AB} = \rho_A \otimes \rho_B \iff S(AB) = S(A) + S(B) \tag{4.13}$$

$$\rho_{AB} \neq \rho_A \otimes \rho_B \iff S(AB) < S(A) + S(B) \tag{4.14}$$

The von Neumnn entropy has a few enlightening properties known as *strong subadditivity*

$$S(AB) + S(BC) \geq S(B) + S(ABC) \tag{4.15}$$

It also obeys the *Araki-Lieb inequality*

$$S(AB) \geq |S(A) - S(B)| \tag{4.16}$$

and *weak monotonicity*

$$S(AB) + S(BC) \geq S(A) + S(C) \tag{4.17}$$

A very useful property for our purposes will be that when AB is in a pure state, then this means

$$S(A) = S(B) \tag{4.18}$$

4.3 Ideas from Quantum information

We will start with the setup of (reference equation) with a bipartate system and a pure state on it.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.19}$$

We have clearly started out with a pure state as it cannot be factorised. Our notation is such that $|00\rangle = |0\rangle_A \otimes |0\rangle_B$ and so on. By taking the partial trace over this we can compute the marginal states.

$$\psi_A = \mathrm{Tr}_B\,\psi = \frac{1}{2}(|0\rangle_A \langle 0| + |1\rangle_A \langle 1|) \tag{4.20}$$

$$\psi_B = \mathrm{Tr}_A\,\psi = \frac{1}{2}(|0\rangle_B \langle 0| + |1\rangle_B \langle 1|) \tag{4.21}$$

We have ended up with both being mixed states. We can check this by noting that they are both proportional to the identity operator on the total Hilbert space and checking. $\mathrm{Tr}\big((\frac{I}{2})^2\big) = \frac{1}{2} < 1$ In this example we have started off with a pure state, but have uncovered entanglement in the system. This notion will be useful later when we consider the bipartate system of a black hole interior and its radiation. This also starts its life as a pure state, and appears to move into a mixed state. This is where the paradox arises.

An important property of bipartate systems in a pure state is that the two systems will have equal entropy.

4.3.1 Purifiying mixed states

This will be important later to understanding the sudden reduction in the fine grained entropy once the black hole has reduced to half of it's initial mass.

Say we have a mixed state on some system, ρ_A on A. If we so choose, we can add another system, B, onto A, to make a larger hilbert space. If we do this in a specific way, the state on $A \cup B$ will be a pure state, and therefor e have zero entropy. Firstly we express ρ_A as a diagonal matrix.

$$\rho_A = \sum_a p_a \left|a\right\rangle_A \left\langle a\right| \tag{4.22}$$

Where p_a are the eigenvalues of ρ_A. We can now define the pure state $|\psi\rangle$.

$$|\psi\rangle := \sum_a \sqrt{p_a} \left|a\right\rangle_A \otimes \left|a\right\rangle_B \tag{4.23}$$

Where $|a\rangle_B$ are an orthonormal set of vectors on $\mathcal{H}_B$.

4.4 The replica trick

Our ultimate goal here will be to arrive at a formula for the entropy of an evaporating black hole. A helpful result will be the EE of a single interval in 1+1 CFT. To do this we use the replica trick. We arrive at this method by the calculation of Renyi

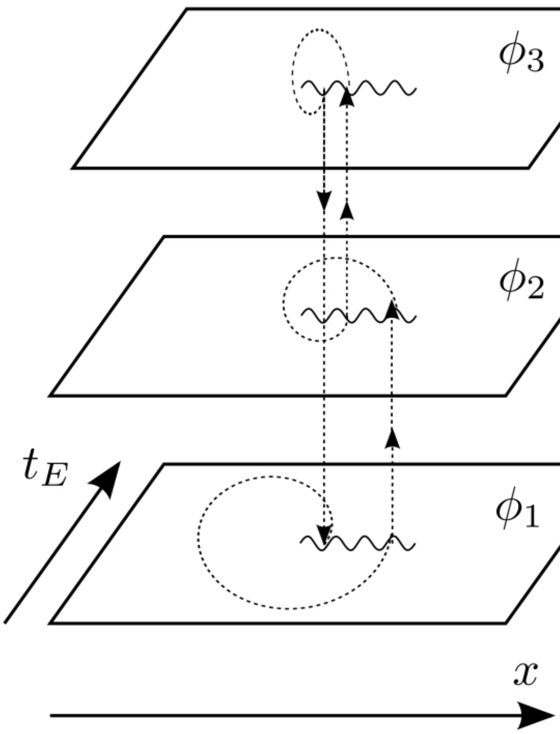

Figure 4.1: Representation of our α sheeted surface glued together along our subregion [23].

entropies [20, 22, 23, 24]

$$S = -\operatorname{Tr}\rho\ln\rho = -\frac{\partial}{\partial n}\operatorname{Tr}\rho^n|_{n=1} \tag{4.24}$$

When $\rho = |\psi\rangle\langle\psi|$ We view this as α copies of the original sheet and glue them together at the boundaries, eventually connecting them cyclically at the boundaries [25].

$$\langle\phi_0^A|\,\rho_A^2\,|\phi_2^A\rangle = \int D\phi_1^A\,\langle\phi_0^A|\,\rho_A\,|\phi_1^A\rangle\,\langle\phi_1^A|\,\rho_A\,|\phi_2^A\rangle \tag{4.25}$$

Where we have inserted a complete set of states in the middle to integrate over. To link this two sheeted surface, we take the trace by setting $\phi_0^A = \phi_2^A$. Fig 4.2 is a representation of our surface that we integrate over in order to find the Rényi entropies.

The calculation first done by Holzhey et al. [26] finds that $\operatorname{Tr}\rho_A^n$ are given by (noting that $\operatorname{Tr}\rho_A = 1$)

$$\operatorname{Tr}\rho_A^n = (Z_1)^{-n} \int_{(t_E,x)\in\mathcal{R}_n} \mathcal{D}\phi e^{-S(\phi)} = \frac{Z_n}{(Z_1)^n} \tag{4.26}$$

The calculation of equation (4.26) is given in the language of CFTs by evaluating two point functions of twist operators. One shows that the α'th Rényi entropy is given by

$$S_\alpha = \frac{c}{6}(1+\frac{1}{\alpha})\ln\frac{L}{\epsilon} + finite \tag{4.27}$$

Which for $\alpha = 1$ is the even simpler

$$S(A) = \frac{c}{3}\ln\frac{L}{\epsilon} \tag{4.28}$$

4.5 The page curve

We now know that we can split up any quantum system into a product of two (or more) subsystems. From this we can calculate the entanglement entropy of each individual system by tracing over the total density matrix ρ to find the reduced density matrix $\rho_A = \operatorname{Tr}_B \rho$. We can then go ahead and calculate the von Neumann entropy as $S_A = -\operatorname{Tr}\rho_A \ln\rho_A$.

4.5.1 General quantum systems

Before applying anything to black holes, we can think in a much more general sense about quantum systems. In 1993 Don Page conjectured [3] that if a system of Hilbert space dimension mn is in a random pure state, the average entropy of a subsystem of dimension $m \leq n$ is $S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$. This was building upon earlier work by Lubkin [27]. He found that for a random pure state of some system AB, with $A < B$. The smaller system, A, will be essentially maximally mixed. This means that the entanglement entropy of that system will be nearly maximal and therefore near our upper bound of entropy, $\ln \dim \mathcal{H}_A$. So the entropy will follow the simple formula.

$$S(A) = \text{min}\{\ln \dim \mathcal{H}_A, \ln \dim \mathcal{H}_B\} + \mathcal{O}(1) \tag{4.29}$$

If we give the system a large number of degrees of freedom, we get a page curve for the entropy of the subsystem A. We will now see this applied to black holes.

4.5.2 Page curve applied to black holes

Due to Page [3, 4] we can view a black hole in this way by splitting the system up into a black hole region and the radiation. Since we know that a black hole is formed from a pure state, we know that $S_{VN}(\mathcal{H}_{BH} \cup \mathcal{H}_R) = 0$. Where we split up the total Hilbert space into a black hole region and the radiation. Page argues that the entropy of the radiation grows linearly with time from zero (since we have a pure state). At this time, we will have $S_{VN}(\mathbf{R}) = S_{rad}$ and hence creates entropy out of nowhere. This is where our problem lies. Page argues that the von Neumann entropy of the radiation increases at early times, but after a so called ”Page time”, the degrees of freedom in the radiation become more entangled with degrees of freedom inside the black hole. We now have $S_{VN}(\mathbf{R}) = S_{BH}$

Following this logic we can derive the page curve for the Schwarzschild black hole. In [3] Page states that in a bipartate system $\mathcal{H} = A \otimes B$ where $|A| << |B|$, the entropy of system A is essentially thermal, meaning $S(A) \approx \ln(\text{dimension of}\mathcal{H}_A)$

We know that the black hole will evaporate in some finite time t_e. The fine grained entropy of the radiation will increase with time up until the page time t_{page}. We know that the BH emmits a thermal spectrum of hawking radiation and we have some number N_f species of massless particles. We also know that all of our quantities are time dependent so we have time dependent temperature $T(t)$, mass $M(t)$, BH and radiation entropy $S_{BH}(t)$ and $S_{\text{rad}(t)}$ and we also know black hole thermodynamics

$$T = \frac{1}{8\pi M} \tag{4.30}$$

$$dM = TdS_{BH} \tag{4.31}$$

$$\frac{dM}{dt} = -N_f \frac{\pi T^2}{12} \tag{4.32}$$

Now we go forth with these relations and solve for our variables.

$$\frac{dT}{dt} = \frac{2\pi^2 N_f T^4}{3} \implies T(t) = \frac{1}{(2\pi^2 N_f)^{\frac{1}{3}}(t_e - t)^{\frac{1}{3}}} \tag{4.33}$$

And

$$M(t) = \frac{1}{8\pi T} \tag{4.34}$$

We can now derive the BH entropy over time. Again solving simple ODEs, we begin with equation (4.31)

$$\frac{dS_{BH}}{dt} = \frac{1}{T}\frac{dM}{dt} \tag{4.35}$$

$$\int_t^{t_e} dS_{BH} = \frac{(2\pi^2 N_f)^{\frac{2}{3}}}{24\pi} \frac{3(t_e - t)^{\frac{2}{3}}}{2}\Big|_t^{t_e} \tag{4.36}$$

$$S_{BH}(t) = \frac{(2\pi^2 N_f)^{\frac{2}{3}}}{16\pi}(t_e - t)^{\frac{2}{3}} \tag{4.37}$$

$$\tag{4.38}$$

Now we need the entropy of the radiation over time. By similar methods

$$S_{rad} = \frac{\pi N_f}{4(2\pi^2 N_f)^{\frac{1}{3}}}(t_e^{\frac{2}{3}} - (t_e - t)^{\frac{2}{3}}) \tag{4.39}$$

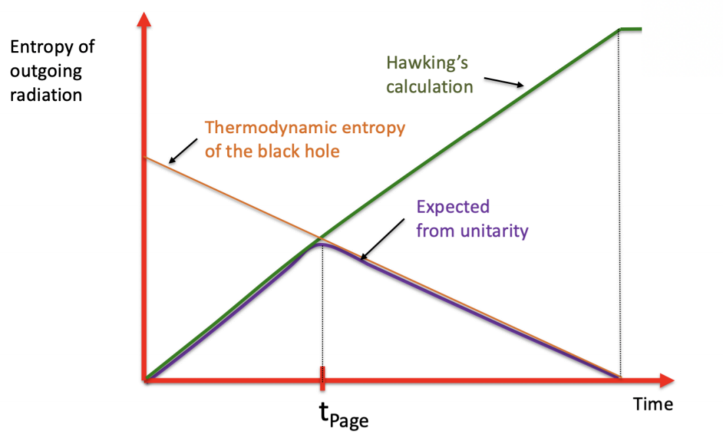

Figure 4.2: The page curve, figure taken from[29]

So S_{rad} grows from 0 and increases. Using pages argument mentioned before, at early times we take the von Neumann entropy to be the entropy of the radiation, at the page time we replace this with the Bekinstein-Hawking entropy. so our formula can be expressed as [4, 28].

$$S_{VN}(\mathbf{R}) = \min\{S_{rad}(t), S_{BH}(t)\} \tag{4.40}$$

Chapter 5

The island rule

So far we have reviewed some necessary material for calculating entanglement entropies. We would like to reproduce the page curve, in calculating the entropy of an evaporating black hole. Hawking's original calculation seems to indicate a loss of information from the universe (Or a creation of entropy). Since the BH starts out in a pure state, which has zero entropy, after evaporation there should still be zero entropy. Another way of saying this is that the S-matrix is unitary.There has been recent progress in calculating the page curve for the black hole [30, 31, 32]. This chapter will aim to outline how we arrive at this rule for calculating the fine grained entropy of the black hole and how this possibly resolves the paradox by reproducing the page curve.

5.1 A word on the AdS/CFT correspondence

Discovered by Maldacena in 1997 [33], he conjectured that certain *Conformal field theories* can be represented as living on the boundary of a gravitational theory in one higher dimension. For example, CFT_2 living on the boundary of AdS_3. We refer to the fields in the AdS theory as *bulk* fields and the CFT fields as $boundary$ fields. We will be using results from this in order to understand the potential resolution of the information paradox. Although it is good to note that the AdS/CFT correspondence gives us the answer to the problem, information is not lost. But having the answer doesn't help us understand why this is true. We will have to do some more work.

5.1.1 Ryu-Takayangi formula

Using the AdS/CFT correspondence we can calculate the holographic entanglement entropy. As we have seen from 2.2.1 the near horizon limit for the extremal Reissner-Nördstrom black hole has an appearence of $AdS_2 \times S^2$. In their paper [34], they propose that the HEE can be calculated by finding "minimal surfaces" on the boundary of AdS. We work in the following setup. The entanlement entropy of a subsystem S_A of a CFT_{d+1} on $\mathbb{R}^{1,d}$. Where the subsystem has a boundary $\partial A \in \mathbb{R}^d$. The CFT_{d+2} lives on the boundary of AdS_{d+2}[34, 35]

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}} \tag{5.1}$$

Where γ_A is our d-dimensional "minimal surface" on the static (not time dependent) slice of AdS and $G_N^{(d+2)}$ is Newtons constant in the dimension of the CFT. One can motivte this formula in many ways, the first being that it looks much like the Bekingstein-Hawking formula for a black hole.

This statement makes calculating entanglement entropies significantly simpler in certain holographic theories. Using the conjeture we are able to reproduce the result for a 1+1 CFT with relative ease. Our problem of calculating the EE of an interval $x \in [-\frac{L_A}{2}, \frac{L_A}{2}]$. Is reduced to finding geodesics of the dual geometry AdS_3. One more simplification we make is to consider a static state, so set $t = 0$. Our metric is now

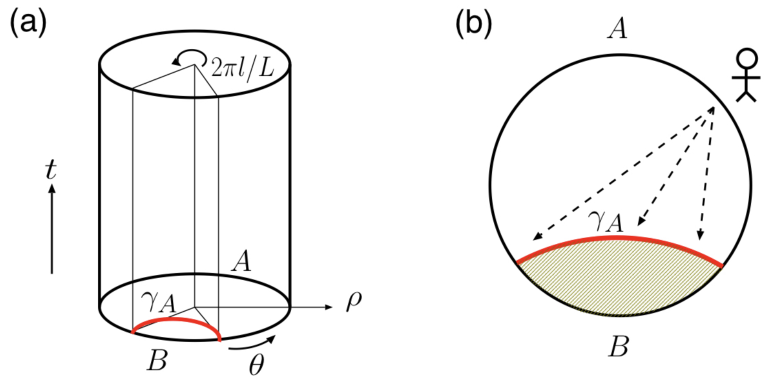

Figure 5.1: The arc in red shows the geodesics on the boundary of AdS_3

[19].

$$ds^2 = \frac{l^2}{z^2}(dx^2 + dz^2) \tag{5.2}$$

Where we must connect the points in our CFT that both lie on the boundary of our geometry, via a geodesic. Once we find the length to be $Length = 2L_A \log\left(\frac{L_A}{\epsilon_{UV}}\right)$ where ϵ_{UV} is our UV cutoff. We plug this into the RT formula, changing paramters between the gravity and the CFT so $L_A = \frac{2cG_N}{3}$ we end up with the expected result

$$S_A = \frac{c}{3}\log\left(\frac{L_A}{\epsilon_{UV}}\right) \tag{5.3}$$

as it is simply a geometry problem. We have the CFT_2 living on the boundary on AdS_3. The minimal surface in the gravity theory is an arc as shown in figure 5.1.

There were subsequent generalisations made by HRT relaxing some of the limitations of the RT formula [36]. They proposed a covariant version of the formula where the entropy of a general boundary region is given by the area of the minimal bulk extremal spacelike surface homologous to A. The difference here being extremal rather than just minimum. So there may be many surfaces that satisfy our condition, we pick the one with the minimum area. The homologous condition is there to make sure the minimal surface is not just the empty set. Later on the HRT definition of the

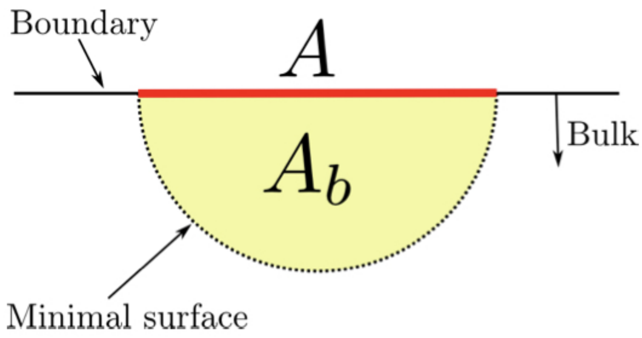

Figure 5.2: This figure shows the CFT living on the boundary of the bulk. A_b is the region inside enclosed by the minimal surface. Figure from [6]

HEE is shown by Wall to be equivalent to a ”maximin surface” prescription [37].

Corrctions of order G_N^0 to this result were found in by Faulkner, Lewkowcyz and Maldacena (FLM)[6]. The correction is given by the von Neumann entropy of the region not contained within the minimal surface in the bulk. Figure 5.2 illustrates this.

$$S(A) = S_{cl} + S_{\text{bulk}}(A_b) \tag{5.4}$$

This was then further generalised to include a perscription of a Quantum extremal surface by Engelhardt and Wall in [7].

$$S_{\text{gen}}(B) = ext_Q\left[\frac{\text{Area}(Q)}{4G_N} + S_{matter}(B)\right] \tag{5.5}$$

The main idea of the EW prescription is that the entropy of the holographic boundary region is given by the generalised entropy of some *quantum extremal surface* rather than on the horizon. B represents the region between the QES and the AdS boundary, similar to the FLM prescription. The quantum extremal surface will be one which extremises S_{gen}, in general there will be many that satisfy this condition. We then pick the one that minimises the entropy. later work then also showed that the maximin prescription applied to the FLM formula was equivalent to finding HRT surfaces.

In order to reproduce the page curve the conjectured ”Island rule” is proposed.

$$S(\rho_R) = \min_Q \left\{ \mathrm{ext}_Q \left[\frac{\mathrm{Area}(Q)}{4G_N} + S(\tilde{\rho}_{I \cup R}) \right] \right\} \tag{5.6}$$

At early times, the radiation is collected in region R and the entropy increases linearly, as expected. If we considered no island, it would continue to increase with no change until the BH had fully evaporated. After the page time, the degrees of freedom in region R start to become entangled with a region known as the ”Island” inside the black hole. Taking this into account and taking the full Cauchy slice to include islands, the entropy starts to decrease in accordance with the Page curve. This can be understood as the states outside being purified with their entanglement pair behind the horizon as we saw in section 4.3.1.

5.1.2 The entanglement wedge

Temporarily forgetting about islands, we turn to entanglement wedge reconstruction. For a given CFT we can define a region in the bulk called the entanglement wedge. As defined in [38], we first make some definitions [39]. The future an past *domain of dependence* of A $D^+(A)$ and $D^-(A)$ respectively are the regions which must be causally influenced by or influence evnts in A. The *causal wedge* $\mathcal{C}_A$ of a subregion A on some boundary Cauchy slice Σ is simply the intersection in the bulk between of the future and past domain of dependence. This picture is illustrated in figure 5.3. The *entantlement wedge* [37, 40, 41] , $\mathcal{E}_A$, is the domain of dependence of some surface in the bulk who's boundary is $A \cup \gamma_A$. Where γ_A is the HRT surface of A. Now why is this at all important? We would like to calculate the fine grained entropy

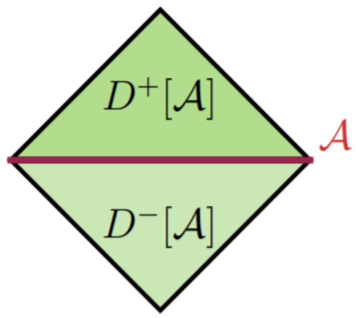

Figure 5.3: The causal wedge $\mathcal{C}_A$ of a subregion A in the CFT in the bulk. figure from [39]

of the Hawking radiation, and show that it reproduces the page curve, this was done using entanglement wedge reconstruction in [30, 31] published simultaneously.

5.2 A useful toy model

In this section we will briefly review the model of a black hole used in a recent paper by Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini (AHMST) [5]. The most useful model for our sake turns out to be two-dimensional Jackiw-Teitelboim (JT) dilaton gravity theory coupled to a matter CFT.

5.2.1 JT gravity

The action for JT gravity + CFT goes as follows

$$I = I_{JT}(\phi, g) + I_{CFT}(g) \tag{5.7}$$

With

$$I_{JT}(\phi, g) = -\frac{\phi_0}{8\pi G_N}\left[\frac{1}{2}\int_{M_2} R + \int_{\partial M_2} K\right] - \frac{1}{8\pi G_N}\left[\frac{1}{2}\int_{M_2} \phi(R+2) + \phi_b \int_{\partial M_2} (K-1)\right] \tag{5.8}$$

The second part of this equation is interesting as we have an interaction between the scalar field (dilaton) and the metric. The CFT part of the action is a general CFT that is not coupled to the dilaton, only the background metric g. Our setup includes an AdS_2 JT gravity region in the middle, glued to a 2-dimensional CFT bath on each side. We impose transparent boundary conditions in order to have matter fields smoothly cross the horizon. We argue that this is a good model for the maximally extended Schwarzschild black hole.

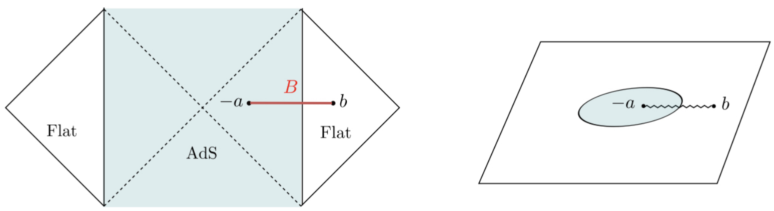

Figure 5.4: Our toy model of a black hole. The left figure shows the Lorentzian signature and the right is Euclidean [5].

In their paper [5], AHMST consider the this toy model of a black hole and calculate the fine grained entropy of the regions discussed directly from a gravity calculation. They take the model for a black hole through a series of conformal transformations with the aim of calculating the entanglement entropy of a region $[-a, b]$ which stretches from the flat CFT region to the AdS region.

To reproduce the Page curve, AHMST use the replica method to compute von Neumann entropy of a region R in the flat wedges. We use a Euclidean path integral and glue the copies together along the cuts on R. To get the so-called "Replica wormholes" we fix the conditions on the AdS boundary. Radiation is collected in the CFT bath region and the entropy increases, up to the page time. After the page time, island regions in the black hole interior are included and the curve levels out. This model is of course a non evaporating black hole, so we would not expect the entropy to fall, this resolves this simple version of the paradox.

Chapter 6

Conclusion

In this dissertation I have attempted to give a general reader with a background in theory, some of the tools to understand and see how the paradox could be solved. To gain an even better understanding, we turn to the explicit computations in the CFT language. We saw how the replica trick can be used to calculate the entanglement entropy of a subsystem and the implications of it in the black hole context.

Here we have set out just one way that the Page curve has been derived from a simple model of a black hole. Since these calculations were done there is now much more attention on this approach to the black hole information problem. There are new papers being published regularly on this topic such as [42]. Where Goto, Hartman and Tajdini investigate the replica wormholes present in these calculations in greater detail, and for a non-eternal, evaporating black hole. There is of course more work to be done here, but this recent work has found ways to reduce the late time entropy of the fields outside the black hole and produce unitary evolution which is consistent with quantum mechanics. There is now the space and time for attempts to be made at more complicated models and deeper investigations into the quantum properties of black holes.

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